The geometry of Gaussian fields
Anne Estrade [1]

Abstract. The short title of the lecture has to be understood as Some geometric properties of Gaussian random fields. There will be four chapters. First chapter will be dedicated to general definitions and properties of Gaussian fields indexed by the Euclidean space $\mathbb{R}^d$ (with $d \geq 2$). In particular, the crucial role of the covariance function will be emphasized and the special cases of stationary fields or fields with stationary increments will be mentioned. We will also look at stationary Gaussian fields from various points of view : spectral representation, random wave, synthesis and modelization. Second chapter will deal with a geometric feature that is really specific to the multivariate context: anisotropy. We will present various models of Gaussian fields whose distributions all share the property of not being invariant under rotations. We will try to understand which characteristics of the field are impacted by the anisotropy property and how they are impacted. In the third chapter, we will establish Rice formulas. They consist in writing moments of some geometric functionals as integrals of conditional expectations that can be explicitely expressed through Gaussian computations. The considered functionals are those who depend on the level sets of the Gaussian field. These formulas are perfect tools to study both qualitative and quantitative geometric properties of stationary Gaussian fields. In last chapter, we will visit recent works on related topics that open new perspectives in link with spatial statistics, number theory, image analysis or cosmology.

1 Gaussian fields

references for general facts on Gaussian fields [1, 7]
most of the following material can be found in the introductory part of Julie Fournier's PhD dissertation [22]

1.1 definitions

covariance function, regression formula

1.2 stationarity

Bochner theorem, spectral measure, stationary increments

1.3 regularity

formulas for the covariance function of derivatives of the field

1.4 random wave model

[20] for the model, inspired by [26, 11, 5, 6], link with (S)PDE [10, 17]

[1] MAP5, UMR CNRS 8145, Université Paris Descartes - anne.estrade@parisdescartes.fr

2 Isotropy/anisotropy

for a survey on anisotropic models [3]

2.1 anisotropic random waves

Berry's random wave and favorite orientation [20] Prop.3.8, directional statistics [27, 25]

2.2 anisotropic fractional fields

anisotropic Gaussian fractional fields [16], simulation [14], Radon transform and X-ray images [16, 15, 31], fractal dimension results [33, 34]

3 Rice formulas

references for general results [1, 7, 12, 22]

3.1 case $d = d'$

area formula, mean number of critical points, modified Euler characteristic [1], second factorial moment, repulsion for critical points [8]

3.2 case $d > d'$

length of nodal lines in the planar case, effect of anisotropy [20] Prop.3.4

4 Minkovski functionals of excursion sets

definition of Minkovski functionals, case of dimension 2, Euler characteristic, non local functionals [9]

4.1 Gaussian kinematic formula

[1, 2]

4.2 Central Limit Theorems and inference statistics

[21, 24, 19, 13], application in cosmology [30]

4.3 Berrys cancellation

[10, 29], similar problem on the 2-dim torus with link to number theory [32, 28]

4.4 deformed Gaussian fields

"from texture to shape" [18, 23]

References

[1] Adler R.J., Taylor J.E. (2007). *Random Fields and Geometry*, Springer, New York.

[2] Adler R.J., Taylor J.E. (2011). *Topological complexity of smooth random functions*, Ecole d'été de Probabilités de Saint-Flour XXXIX-2009, Lecture Notes in Mathematics 2019, Springer.

[3] Allard D., Senoussi R., E. (2016). Anisotropy models for spatial data. *Mathematical Geosciences*, **48** (3), 305–328.

[4] Azaïs J-M., León, J.R. (2019). Necessary and sufficient conditions for the finiteness of the second moment of the measure of level sets. *Preprint* ArXiv:1905.12342

[5] Azaïs J-M., León, J.R., Ortega, J. (2005). Geometrical Characteristics of Gaussian Sea Waves. *Journal of Applied Probability*, **42,** 407–425.

[6] Azaïs J-M., León J.R., Wschebor M. (2011). Rice formula and Gaussian waves. *Bernoulli*, **17**, 170–193.

[7] Azaïs, J-M., Wschebor M. (2009). *Level Sets and Extrema of Random Processes and Fields.* Wiley.

[8] Beliaev D., Cammarota V., Wigman I. (2019). Two point function for critical points of a random plane wave. *International Mathematics Research Notices*, **9**, 2661-2689.

[9] Beliaev D., McAuley M., Muirhead S. (2018). On the number of excursion sets of planar Gaussian fields. *Preprint* arXiv:1807.10209

[10] Berry, M.V. (2002). Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. *Journal of Physics A*, **35**, 3025–3038.

[11] Berry M.V., Dennis M.R. (2000). Phase singularities in isotropic random waves. *Proc. R. Soc. Lond. A* **456**, 2059-2079.

[12] Berzin C., Latour A., León J.R. (2017). *Kac-Rice formulas for random fields and theirs applications in: random geometry, roots of random polynomials and some engineering problems.* Ediciones IVIC, Instituto Venezolano de Investigaciones Científicas, Caracas.

[13] Biermé H., Di Bernardino E., Duval C., Estrade A. (2019) Lipschitz-Killing curvatures of excursion sets for two dimensional random fields, *Electron. J. Statist.* **vol 13**; 536-581.

[14] Biermé H., Moisan L., Richard F. (2015). A turning band method for the simulation of anisotropic fractional Brownian fields. *J. Comp. and Graph. Stat.*, **vol 24** (3), 885–904.

[15] Biermé H., Richard F. (2008). Estimation of anisotropic Gaussian fields through Radon transform. *ESAIM Probab. Stat.*, **12** (1), 30–50.

[16] Bonami A., Estrade A. (2003). Anisotropic analysis of some Gaussian models. *The Journal of Fourier Anal. and Appl.* **9**, 215-236.

[17] Carrizo R., Allard D., Desasis N., (2018). A general framework for SPDE-based stationary random fields. *Preprint* arXiv:1806.04999

[18] Clerc M., Mallat S. (2003). Estimating deformations of stationary processes. *the Annals of Stat.* **31** (6), 1772-1821.

[19] Di Bernardino E., Estrade A., Leon J.R., A test of Gaussianity based on the Euler characteristic of excursion sets, *Electron. J. Statist.* **vol 11**; 843-890.

[20] Estrade A., Fournier J. (2018). Anisotropic Gaussian wave models. *Preprint.* hal-01745706.

[21] Estrade A., Leon J.R. (2016), A central limit theorem for the Euler characteristic of a Gaussian excursion set, *Ann. Prob.* **vol 44** (6); 3849–3878.

[22] Fournier J. (2018). Geometric characteristics of smooth anisotropic random fields. *PhD Dissertation*, https://tel.archives-ouvertes.fr/tel-02155827

[23] Fournier J. (2019). Identification and isotropy characterization of deformed random fields through excursion sets. *Advances in Applied Probability.* **50** (3).

[24] Kratz M., Vadlamani S. (2017). Central Limit Theorem for Lipschitz–Killing Curvatures of Excursion Sets of Gaussian Random Fields. *J. Theor. Probab.* **26** (3), 1–30.

[25] Ley C., Verdebout T. (2017). *Modern directional statistics*, CRC Press.

[26] Longuet-Higgins M.S. (1957). The Statistical Analysis of a Random, Moving Surface. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* **249**, 321–387.

[27] Mardia K.V., Jupp P.E. (2009). *Directional statistics*, John Wiley & Sons.

[28] Marinucci D., Peccati G., Rossi M., Wigman I. (2016). Non-Universality of Nodal Length Distribution for Arithmetic Random Waves. *Geometric and Functional Analysis*, **3**, 926–960.

[29] Nourdin I., Peccati G., Rossi M. (2019). Nodal statistics of planar random waves. *Comm. Math. Phys.* **369**(1), 99–151.

[30] Planck (2015). Planck 2015 results XVI. Isotropy and statistics of the CMB. Astronomy and Astrophysics. http://arxiv.org/abs/1506.07135

[31] Richard F., Biermé H. (2010). Statistical tests of anisotropy for fractional Brownian textures. Application to full-field digital mammography. *J. Math. Imaging and Vision*, **vol 36** (3), 227–240.

[32] Rudnick Z., Wigman I. (2008). On the volume of nodal sets for eigenfunctions of the Laplacian on the torus. *Ann. Henri Poincaré*, **9** (1), 109–130.

[33] Xiao Y. (2009) Sample path properties of anisotropic Gaussian random fields. *In: A Minicourse on Stochastic Partial Differential Equations, (D. Khoshnevisan and F. Rassoul-Agha, editors), Lecture Notes in Math.* **1962**, pp. 145–212, Springer, New York.

[34] Xiao Y., Xue Y. (2011). Fractal and smoothness properties of space-time Gaussian models. *Frontiers Math. in China*, **6**, 1217–1246.